Bipolar Transistor Circuit Analysis Using the Lambert W-Function

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***Abstract*—The generalized diode equation describes conduction in a diode with series resistance. An analytical solution for the generalized diode equation has been elusive; however, one was found based on the transcendental equation $w = \ln(x/w)$. The solution of this equation; $w = \mathrm{W}(x)$, is traditionally referred to as the Lambert W-function. This function provides a long sought after natural continuity between exponential diode and linear resistor behavior. The W-function also describes more general circuits consisting of a diode or bipolar transistor with local linear negative or positive feedback. The properties of $\mathrm{W}(x)$ are reviewed and several iterative methods for its calculation are compared. Three approximations for the W function are derived which can simplify bipolar circuit analysis and design. The practical utility of the proposed solutions are demonstrated in four circuits along with experimental confirmation: a common emitter amplifier with an emitter or collector feedback resistor, Schmitt trigger threshold temperature compensation, bandgap stabilized current source, and a novel current-efficient laser driver.**

***Index Terms*—Bipolar transistor circuits, circuit analysis, generalized diode equation, nonlinear circuits.**

I. Introduction

SEVERAL authors have noted the difficulty in deriving an analytical solution for the simple circuit shown in Fig. 1. consisting of a diode driven by a voltage source V_{Th} through a series resistor R_{Th} [1], [2]. The current and voltage at the diode terminals are given by simultaneous solution of the two equations

$$I_d = I_{\mathrm{sat}}\left(e^{V_d/V_T} - 1\right) \tag{1}$$

$$V_d(I_d) = V_{Th} - I_d R_{Th} \tag{2}$$

where $V_T = k_B T/q_e$ is the thermal voltage and I_{sat} is the diode's reverse saturation current. The solution of these equations is sometimes referred to as the "generalized diode equation." It is common practice to evaluate (1) and (2) using a general numerical method such as the Newton–Raphson iteration [3]. Several approximate analytical solutions have also been reported [4]–[6]. It is well known that an explicit analytical solution for I_d cannot be constructed with just the common elementary functions [2]. However, the earlier work does not preclude the existence of an answer in terms of a possibly less familiar function. As a first step toward the desired solution, (1) and (2) are combined by eliminating V_d. The resulting expression can

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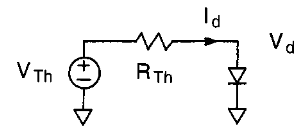

Fig. 1. Ideal diode driven by voltage source V_{Th} through series resistance R_{Th}.

then be factored to separate the dependent and independent variables. This particular arrangement of terms leads to

$$\frac{(I_d + I_{\mathrm{sat}})R_{Th}}{V_T}\, e^{(I_d+I_{\mathrm{sat}})R_{Th}/V_T} - \frac{I_{\mathrm{sat}} R_{Th}}{V_T}\, e^{(V_{Th}+I_{\mathrm{sat}}R_{Th})/V_T} = 0. \tag{3}$$

The left-hand side of (3) is a function of the diode current I_d through the single quantity $w = (I_d + I_{\mathrm{sat}})R_{Th}/V_T$. Rewriting (3) in terms of w results in the simple transcendental equation

$$we^w - x = 0 \tag{3a}$$

where x represents the second term in (3) containing the independent variable V_{Th}. There are several theorems which assert that the desired solution exists [7]. Equation (3a) is a continuous function of w and has a bounded and nonvanishing first derivative with respect to w for finite $w \geq 0$. One can conclude from the implicit function theorem, for example, that (3) and (3a) have a unique inverse for $x \geq 0$. The inverse function can be expressed as $w = \mathrm{W}(x)$ and consequently the desired analytical solution for I_d is

$$I_d = -I_{\mathrm{sat}} + \frac{V_T}{R_{Th}}\, \mathrm{W}\left(\frac{I_{\mathrm{sat}} R_{Th}}{V_T}\, e^{(V_{Th}+I_{\mathrm{sat}}R_{Th})/V_T}\right). \tag{4}$$

A careful search of the literature reveals that the solution $w = \mathrm{W}(x)$ for (3) is known as the Lambert W-function, named after J. H. Lambert for work that he published in 1758 [8]. Although this is not yet a common function, its properties are easily derived and well documented [8]–[11]. There are several published algorithms for calculating the W-function and a simple but efficient one is discussed below. Before reviewing the relevant properties of the W-function, it is worth noting first that it also provides an analytical solution for two other common circuits as well.

Many complex circuits consisting of resistors and voltage or current sources can be reduced to a Thevénin equivalent circuit represented by Fig. 1. The Widlar current source in Fig. 2 is a special case of Fig. 1 where V_{BE1} corresponds to V_{Th}. It is extensively employed for bias and temperature compensation [12]. An analytical solution for this well-known circuit has not